The p-adic Mehta Integral: Formulas, Functional Equations, and Combinatorics

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Abstract: The classical *Mehta integral* is defined by

$$Z_N(\beta) = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(x_1^2+\cdots+x_N^2)} \prod_{1\leq i<j\leq N} |x_i - x_j|^\beta \, dx_1 \ldots dx_N$$

and it serves as the canonical partition function for an N-particle log-Coulomb gas in a 1-dimensional harmonic potential well at inverse temperature $\beta > 0$. Among the first to study it were Mehta and Dyson, who computed its values at $\beta = 1, 2, 4$ to determine the joint probability densities for the eigenvalues of $N \times N$ Gaussian random matrix ensembles. Bombieri later proved a closed form for $Z_N(\beta)$ that is valid for all complex β with $\mathrm{Re}(\beta) > -2/N$. In this talk we will introduce the *p-adic Mehta integral* as a canonical partition function for an analogous p-adic log-Coulomb gas and offer a new method for finding its explicit value and domain. We will discuss how this method generalizes easily to multicomponent log-Coulomb gases in arbitrary nonarchimedean local fields and their projective lines, and how it unites them all under a common combinatorial framework. Using a quadratic recurrence, we will also produce efficient computational alternatives, the $p \to 1$ limits, and the $p \mapsto p^{-1}$ functional equations for the p-adic Mehta integral and its projective analogue.